Roots of a Type of Generalized Quasi-Fibonacci Polynomials

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Abstract

Let a be a nonnegative real number and define a quasi-Fibonacci polynomial sequence by $F_1^a(x) = -a, F_2^a(x) = x - a$, and $F_n^a(x) = F_{n-1}^a(x) + xF_{n-2}^a(x)$ for $n \geq 2$. Let r_n^a denote the maximum real root of F_n^a. We prove for certain values of a that the sequence $\{r_{2n}^a\}$ converges monotonically to $\beta_a = a^2 + a$ from above and the sequence $\{r_{2n+1}^a\}$ converges monotonically to β_a from below.

1 Introduction

Consider a Fibonacci type polynomial sequence defined recursively by $G_0(x) = \alpha$, $G_1(x) = x + \beta$, and for $n \geq 2$

$$G_n = xG_{n-1} + G_{n-2}.$$

Here α and β are integers with $\alpha \neq 0$. The properties of these polynomials have been studied extensively: when $\alpha = 1$ and $\beta = 0$, G_n is the classical Fibonaccci polynomial sequence. Likewise, when $\alpha = 2$ and $\beta = 0$, one gets the classical Lucas polynomials. Hoggatt and Bicknell explicitly find the zeros of these two sequences in [2]. Moore [4] and Prodinger [5] studied the asymptotic behavior of the maximal roots when $\alpha = \beta = -1$. Moore's results have been generalized to more cases; in [6], Yu, Wang, and He study the case when $\alpha = \beta = -a$ for all positive integers a, while Mátyás [3] examines the sequence for $\alpha = a \neq 0$ and $\beta = \pm a$. More recently, Wang and He fully generalized their results to any integers α and β with $\alpha \neq 0$.

Let a be a positive real number and define the quasi-Fibonacci polynomial sequence F_n^a recursively by $F_0^a(x) = -a$, $F_1^a(x) = x - a$, and for $n \geq 2$,

$$F_n^a(x) = F_{n-1}^a(x) + xF_{n-2}^a(x). \tag{1}$$

In [1], B. Alberts studies this sequence when $a = 1$ and finds that the maximum roots of even indexed polynomials converge monotonically to 2 from above, while the maximum roots of the odd indexed polynomials converge monotonically to 2 from below. In general, denote the maximum root of $F_n^a(x)$ by r_n^a . When no ambiguity will arise we will omit the superscript ^{a}.

The paper is organized as follows: in section 2 we present technical results used to prove later theorems. In section 3 we examine the existence, boundedness, and monotonic behavior of the maximum roots. In sections 4 and 5 we study properties of the derivative needed to show convergence. Finally, main results are provided in section 6.

2 Technical results

Lemma 1. *Denote the standard Fibonacci sequence by $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Then*

$$F_n(1) = f_n - a \cdot f_{n+1} \tag{2}$$

$$F_n(0) = -a \tag{3}$$

$$F_n(-1) \in \{-a, -1 - a, -1, a, 1 + a, 1\} \tag{4}$$

Proof. The proofs for Equations (2) and (3) follow directly from the initial conditions and recursion. For Equation (4), we notice that $F_0(-1) = -a$, $F_1(-1) = -1 - a$, $F_2(-1) = -1$, $F_3(-1) = a$, $F_4(-1) = 1 + a$, $F_5(-1) = 1$. An inductive argument shows this pattern holds for all values of n.

Lemma 2. *Let $\beta = a^2 + a$. Then $F_n(\beta) = (-a)^{n+1}$.*

Proof. This is clear for F_0 and F_1. Suppose it holds through $n - 1$. Then

$$\begin{aligned} F_n(\beta) &= F_{n-1}(\beta) + \beta \cdot F_{n-2}(\beta) \\ &= (-a)^n + \beta(-a)^{n-1} \\ &= (-a)^{n-1}(-a + \beta) = (-a)^{n-1}(a^2) = (-a)^{n+1}. \end{aligned}$$

Lemma 3.

$$F_n(x) = (1+2x)F_{n-2}(x) - x^2 F_{n-4}(x) \tag{5}$$

Proof. This follows directly from manipulation of the recursive formula.

Theorem 4.

$$F_n(x) = \sum_{i=0}^{\infty} \left[\binom{n-i}{i-1} - a\binom{n-i}{i} \right] x^i \tag{6}$$

Proof. Notice that each sum is finite since for large enough i, $i-1 > n-i$ and $i > n-i$. Direct computation verifies the formula for F_0 and F_1; suppose we have shown the theorem through F_{n-1}. Then

$$\begin{aligned}
F_n(x) &= F_{n-1}(x) + xF_{n-2}(x) \\
&= \sum_{i=0}^{\infty} \left[\tbinom{n-1-i}{i-1} - a\tbinom{n-1-i}{i}\right] x^i + x \cdot \sum_{i=0}^{\infty} \left[\tbinom{n-2-i}{i-1} - a\tbinom{n-2-i}{i}\right] x^i \\
&= \sum_{i=0}^{\infty} \left[\tbinom{n-1-i}{i-1} - a\tbinom{n-1-i}{i}\right] x^i + \sum_{j=1}^{\infty} \left[\tbinom{n-1-j}{j-2} - a\tbinom{n-1-j}{j-1}\right] x^j + 0 \\
&= \sum_{i=0}^{\infty} \left[\tbinom{n-i}{i-1} - a\tbinom{n-i}{i}\right] x^i
\end{aligned}$$

Corollary.

$$F_{2n}(x) = \sum_{i=0}^{n} \left[\binom{2n-i}{i-1} - a\binom{2n-i}{i} \right] x^i \tag{7}$$

$$F_{2n+1}(x) = \sum_{i=0}^{n+1} \left[\binom{2n+1-i}{i-1} - a\binom{2n+1-i}{i} \right] x^i \tag{8}$$

3 Positive real roots

Lemma 5.

If $\binom{n-i}{i-1} - a \cdot \binom{n-i}{i} > 0$, *then* $\binom{n-(i+1)}{(i+1)-1} - a \cdot \binom{n-(i+1)}{(i+1)} > 0$.

If $\binom{n-i}{i-1} - a \cdot \binom{n-i}{i} \leq 0$, *then* $\binom{n-(i-1)}{(i-1)-1} - a \cdot \binom{n-(i-1)}{(i-1)} < 0$.

Proof. Suppose $\binom{n-i}{i-1} - a \cdot \binom{n-i}{i} > 0$. Then $\frac{i}{n-2i+1} = \frac{\binom{n-i}{i-1}}{\binom{n-i}{i}} > a$. So

$$\frac{\binom{n-(i+1)}{(i+1)-1}}{\binom{n-(i+1)}{(i+1)}} = \frac{(i+1)}{n-2(i+1)+1} > \frac{i}{n-2i+1} > a$$

Our conclusion follows directly from this fact; the second result is proven similarly.

Theorem 6. *F_{2n+1} has exactly one nonnegative real root. When $n \leq a$, F_{2n} has no positive real roots, and when $n > a$, F_{2n} has exactly one nonnegative real root.*

Proof. First notice that if $a = 0$ then the maximal root of each F_n is 0 for $n \geq 1$.

Suppose $a > 0$. We use Descartes' Rule of Signs. Equation (8) shows that the constant term of F_{2n+1} is $-a < 0$ and the leading coefficient of F_{2n+1} is $1 > 0$. Thus, by Lemma 5, there must be exactly one change in sign of the coefficients of F_{2n+1}. This implies that the polynomial has exactly one real positive root.

Similarly, Equation (7) shows the constant term of F_{2n} is $-a < 0$. However, when $n \leq a$, the leading coefficient of F_{2n} is $n - a \leq 0$, so there are no sign changes and therefore no positive real roots. When $n > a$, $n - a > 0$, so there must be exactly one sign change and therefore exactly one positive real root.

Corollary. *Let r_n denote the maximal real root of the polynomial F_n (if it exists). Then for all n, r_{2n+1} exists and $r_{2n+1} < \beta$.*

Similarly, for all $n > a$, r_{2n} exists and $\beta < r_{2n}$.

Corollary. *For all n and for all $x \in (0, r_{2n+1})$, $F_{2n+1}(x) < 0$; for all $x > r_{2n+1}$, $F_{2n+1}(x) > 0$.*

Similarly, for all $n > a$, for all $x \in (0, r_{2n})$, $F_{2n}(x) < 0$ and for all $x > r_{2n}$, $F_{2n}(x) > 0$.

Theorem 7. *For all n,*

$$a = r_1 < r_3 < \cdots < r_{2n+1} < \cdots < \beta. \tag{9}$$

Furthermore, when $N > a$ and $n > N$,

$$\beta < \cdots < r_{2n} < \cdots < r_{2N+2} < r_{2N}. \tag{10}$$

Proof. Clearly $r_1 = a$ and $F_3(r_1) = F_2(a) < 0$, so $r_1 < r_3$. Suppose we have shown Equation (9) holds through r_{2n-1}. Then since $r_{2n-3} < r_{2n-1}$, Equation (5) gives us

$$\begin{aligned} F_{2n+1}(r_{2n-1}) &= (1 + 2r_{2n-1})F_{2n-1}(r_{2n-1}) - r_{2n-1}^2 F_{2n-3}(r_{2n-1}) \\ &= -r_{2n-1}^2 F_{2n-3}(r_{2n-1}) < 0. \end{aligned}$$

Thus, $r_{2n+1} > r_{2n-1}$.

The proof of (10) is similar.

4 Behavior of derivatives

For the remainder of the paper, we will use N_a (or just N) to denote an integer satisfying the following four properties:

(i) $N_a > a$.

(ii) $r_{2N_a+1} \geq a^2$.

(iii) For $x \geq \beta$, $F'_{2N_a-1}(x) > a^{2N_a-1}$ and $F'_{2N_a}(x) > a^{2N_a}$.

(iv) For $x > r_{2N_a+1}$, $F'_{2N_a+1}(x) \geq 1$, $F'_{2N_a}(x) \geq 1$, and $F'_{2N_a+2}(x) \geq xF'_{2N_a}(x)$.

Numerical evidence suggests that such an N_a exists for all $a \geq 0$. For instance, when $0 \leq a \leq 1$, it can be shown directly that $N_a = 2$ satisfies the properties.

Lemma 8. *Let $x \geq \beta$. Then for all $n \geq 2N - 1$, $F'_n(x) > a^n$.*

Proof. Since N satisfies condition (iii), $F'_{2N-1}(x) > a^{2N-1}$ and $F'_{2N}(x) > a^{2N}$. Suppose we have shown the lemma holds for all integers between $2N-1$ and $2n$. Then since $F'_{2n-1} > 0$ on $[\beta, \infty)$, $F_{2n-1}(x) \geq F_{2n-1}(\beta)$, so

$$\begin{aligned} F'_{2n+1}(x) &= F'_{2n}(x) + xF'_{2n-1}(x) + F_{2n-1}(x) \\ &\geq a^{2n} + \beta \cdot a^{2n-1} + a^{2n} \\ &= a^{2n+1} + 3 \cdot a^{2n} > a^{2n+1}. \end{aligned}$$

The proof for F'_{2n+1} follows similarly.

Lemma 9. *For all $n > N$ and $x > r_{2n-1}$,*

$$\frac{F'_{2n+1}(x)}{x^{n-N}} \geq \frac{F'_{2n-1}(x)}{x^{n-N-1}} \geq \cdots \geq F'_{2N+1}(x) \geq 1$$

and

$$\frac{F'_{2n+2}(x)}{x^{n-N+1}} \geq \frac{F'_{2n}(x)}{x^{n-N}} \geq \cdots \geq F'_{2N}(x) \geq 1.$$

Proof. Let $x > r_{2n-1}$. Since N satisfies condition (iv), $F'_{2N+1}(x) \geq 1$, $F'_{2N}(x) \geq 1$, and $F'_{2N+2}(x) \geq xF'_{2N}(x)$. Suppose we have shown the lemma holds through $n-1$. Let $x \geq r_{2n-1} > r_{2n-3}$. Then

$$\begin{aligned} F'_{2n+1}(x) &= F'_{2n}(x) + xF'_{2n-1}(x) + F_{2n-1}(x) \\ &\geq x^{n-N} + xF'_{2n-1}(x) + 0 \\ &\geq xF'_{2n-1}(x), \end{aligned}$$

so

$$\frac{F'_{2n+1}(x)}{x^{n-N}} \geq \frac{F'_{2n-1}(x)}{x^{n-N-1}}.$$

Similarly, since $F'_{2n}(x) \geq xF'_{2n-2}(x)$, Equation (5) shows:

$$\begin{aligned} F'_{2n+2}(x) &= (1+2x)F'_{2n}(x) - x^2F'_{2n-2}(x) + 2F_{2n}(x) - 2xF_{2n-2}(x) \\ &\geq F'_{2n}(x) + (2x-x)F'_{2n}(x) + 2F_{2n-1}(x) \\ &\geq x^{n-N} + xF'_{2n}(x) + 0 \\ &\geq xF'_{2n}(x), \end{aligned}$$

so

$$\frac{F'_{2n+2}(x)}{x^{n-N+1}} \geq \frac{F'_{2n}(x)}{x^{n-N}}.$$

Remark Given some n_0 that satisfies condition (iii), each $n \geq n_0$ also satisfies condition (iii). Likewise, if n_1 satisfies condition (iv), each $n \geq n_1$ also satisfies condition (iv).

Lemma 10. *When* $x \geq \beta$, $\lim_{n\to\infty} \frac{F'_{2n}(x)}{a^{2n}} = \infty$.

Proof. Let $x \geq \beta$. Notice from Lemma 8 that when $n > N$ and $x \geq \beta$,

$$\begin{aligned}\frac{F'_{2n}(x)}{a^{2n}} - \frac{F'_{2n-2}(x)}{a^{2n-2}} &= \frac{1}{a^{2n}}\left(F'_{2n-1}(x) + xF'_{2n-2}(x) + F_{2n-2}(x) - a^2F'_{2n-2}\right) \\ &\geq \frac{1}{a^{2n}}\left(a^{2n-1} + (\beta - a^2)F'_{2n-2}(x) - a^{2n-1}\right) \\ &\geq \frac{1}{a^{2n}}\left(a \cdot a^{2n-2}\right) = \frac{1}{a}\end{aligned}$$

Then

$$\begin{aligned}\infty &= \lim_{n\to\infty} \sum_{i=N+1}^{n} \frac{1}{a} \\ &\leq \lim_{n\to\infty} \sum_{i=N+1}^{n} \left[\frac{F'_{2i}(x)}{a^{2i}} - \frac{F'_{2i-2}(x)}{a^{2i-2}}\right] \\ &= \lim_{n\to\infty} \left[\frac{F'_{2n}(x)}{a^{2n}} - \frac{F'_{2N}(x)}{a^{2N}}\right] \\ &= \lim_{n\to\infty} \frac{F'_{2n}(x)}{a^{2n}}\end{aligned}$$

Lemma 11. *Let* $\{x_{2n+1}\}$ *be a sequence of numbers satisfying* $x_{2n+1} \geq r_{2n-1}$. *Then*

$$\lim_{n\to\infty} \frac{F'_{2n+1}(x_{2n+1})}{a^{2n+1}} = \infty.$$

Proof. Since N satisfies condition (ii), $r_{2N+1} \geq a^2$. Then for $n > N$, Lemma 9 and a similar argument to the previous proof show that when $x > r_{2n-1} \geq r_{2N+1} \geq a^2$,

$$\frac{F'_{2n+1}(x)}{a^{2n+1}} - \frac{F'_{2n-1}(x)}{a^{2n-1}} \geq \frac{1}{a^{2n+1}}\left(\frac{a^{2n}}{a^{2N}}\right) = \frac{1}{a^{2N+1}}.$$

Thus, by an analogous argument to the previous proof, we find

$$\lim_{n\to\infty} \frac{F'_{2n}(x)}{a^{2n}} = \infty.$$

5 Existence of N_a

We will now investigate the existence of an N_a which satisfies the base cases for each of the proofs in the previous sections. The properties and conditions mentioned below are listed at the beginning of section 4.

Notice that if n_0 satisfies any given condition, then each $n \geq n_0$ also satisfies that condition. Furthermore, direct computation of N_a for some values of a (see table below) suggests that as a increases, the minimum possible value of N_a increases. This implies that the value N_a would work for all values less than or equal to a; e.g., since $N_{25} = 102$ satisfies the conditions when $a = 30$, 102 should satisfy the conditions for all $a \in [0, 30]$.

a	n satisfying (ii)	n satisfying (iii)	n satisfying (iv)	N_a
1	1	2	2	2
2	2	4	4	4
3	4	7	6	7
5	7	12	12	12
10	18	26	28	28
20	44	59	63	63
30	72	95	102	102

Lemma 12. *Suppose there exists some N satisfying condition (iv) for a given value of a. Then there exists $k \geq 0$ such that $N + k$ satisfies condition (iii).*

Proof. By Lemma 9, for all $k \geq 0$ and $x > r_{2(N+k)-1}$,

$$F'_{2(N+k)+1}(x) \geq x^k \qquad \text{and} \qquad F'_{2(N+k)}(x) \geq x^k$$

Let k be large enough such that

$$\frac{(a^2+a)^{k-1}}{(a^2)^{k-1}} \geq a^{2N+1} \qquad \text{and} \qquad \frac{(a^2+a)^k}{(a^2)^k} \geq a^{2N}.$$

Then for all $x \geq \beta$,

$$F'_{2(N+k)-1}(x) \geq x^{k-1} \geq \beta^{k-1} = (a^2+a)^{k-1} \geq a^{2(N+k)-1}.$$

Likewise, for all $x \geq \beta$,

$$F'_{2(N+k)}(x) \geq x^k \geq (a^2+a)^k \geq a^{2(N+k)}.$$

Thus, $N + k$ satisfies condition (iii).

6 Main results

Theorem 13. *Let a be an integer and $n > 1$. If n is odd or greater than $2a$, then r_n is irrational.*

Proof. Suppose r is a rational root of F_n. Then in reduced form, $r = \frac{p}{q}$ where p is a factor of $-a$. So $|p| \leq a$, which means $r \leq a$. However, from Theorem 7 we know $r_1 = a$, $r_{2n+1} > r_1$, and when $\frac{n}{2} > a$, $r_{2n} > \beta > a$.

Theorem 14. *The sequence r_{2n} converges to β monotonically from above, and the sequence r_{2n+1} converges to β monotonically from below.*

Proof. Recall that $F_n(\beta) = (-a)^{n+1}$, $r_{2n} > \beta$ for $n > a$, and $r_{2n+1} < \beta$. By the Mean Value Theorem, for each $n > a$ there exists $c_{2n} \in [\beta, r_{2n}]$ such that

$$\frac{F_{2n}(r_{2n}) - F_{2n}(\beta)}{r_{2n} - \beta} = F'_{2n}(c_{2n}).$$

Then

$$\begin{aligned} \lim_{n\to\infty} F'_{2n}(c_{2n}) &= \lim_{n\to\infty} \frac{-F_{2n}(\beta)}{r_{2n} - \beta} \\ \lim_{n\to\infty} (r_{2n} - \beta) &= a \lim_{n\to\infty} \frac{a^{2n}}{F'_{2n}(c_{2n})} = 0 \\ \lim_{n\to\infty} r_{2n} &= \beta. \end{aligned}$$

By combining a similar argument and Lemma 11, we obtain our second result.

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