Fibonacci Polynomials
A Generalized Recurrence Relation

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1 Introduction

In this paper, we consider a second order recursive sequence of polynomials G_n where $G_0(x) = -1,\ G_1(x) = x - 1$, and

$$G_n(x) = x^k G_{n-1} + x^l G_{n-2}. \tag{1}$$

Most of the time, we will focus on the case where $l = k$, that is, where

$$G_n(x) = x^k \left(G_{n-1}(x) + G_{n-2}(x)\right).$$

In [1] Moore works on the sequence H_n where $k = 1$ and $l = 0$ and shows that the limit of the maximal roots of H_n converge to $\frac{3}{2}$. Specifically, the even indexed maximal roots monotonically decrease to $\frac{3}{2}$ and the odd indexed monotonically increase to $\frac{3}{2}$. He also gives a variety of formulae for $H_n(x)$, including a shift formula and summation formula.

A computer analysis of the roots of the polynomials $G_n(x)$ suggests that the maximal roots converge to 2 as $n, k \to \infty$, with the even indexed roots monotonically decreasing and the odd indexed roots monotonically increasing.

2 Modular Identities

Although there are a lot of fields which study recursive polynomials, we present some modular identities that could be further studied more in detail.

$$G_n(x) \equiv a^k (G_{n-1}(x) + G_{n-2}(x)) \pmod{x - a} \tag{2}$$

Proof. This is very obvious since $x^k \equiv a^k \pmod{x - a}$. □

Note that one can get a closed form for this equality by just replacing x^k by a^k in the Binet's Formula for $G_n(x)$. Also when $a = 1$, we have

$$G_n(x) \equiv -F_{n-1} \pmod{x - 1} \tag{3}$$

where $F_n(x)$ is the n'th fibonacci number.

Proof. Note that $G_1(x) \equiv 0 \pmod{x-1}$. Also $G_2(x) \equiv -1 \pmod{x-1}$. Thus, assume that for all $i < n$ $G_i(x) \equiv -F_{i-1} \pmod{x-1}$. Hence,

$$\begin{aligned} G_n(x) &= x^k(G_{n-1}(x) + G_{n-2}(x)) \\ &\equiv (-F_{n-2} - F_{n-3}) \pmod{x-1} \\ &\equiv -F_{n-1} \pmod{x-1} \end{aligned}$$

□

3 Binet's Formula

The first formula we consider is the 'Binet Formula', which we obtain by using the process of getting the Binet Formula for the Fibonacci numbers.

Formula 2.

$$\begin{aligned} G_n(x) = &\frac{\left[\frac{1}{2}\left(x^k - \sqrt{x^{2k}+4x^k}\right) + x - 1\right]\left(x^k + \sqrt{x^{2k}+4x^k}\right)^n}{2^n\sqrt{x^{2k}+4x^k}} \\ &- \frac{\left[\frac{1}{2}\left(x^k + \sqrt{x^{2k}+4x^k}\right) + x - 1\right]\left(x^k - \sqrt{x^{2k}+4x^k}\right)^n}{2^n\sqrt{x^{2k}+4x^k}} \end{aligned} \tag{4}$$

Proof. Consider an equation of the form

$$a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$$

with $a_0(x)$ and $a_1(x)$ as the initial conditions. Note that the characteristic equation of this recursion is

$$t^2 - p(x)t - q(x) = 0.$$

Denote the solutions by $A(x)$ and $B(x)$. Then

$$A(x) = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2}$$

and

$$B(x) = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}$$

Then, by the process of Binet's Formula, we obtain

$$a_n(x) = \left(\frac{a_1(x) - B(x)a_0(x)}{A(x) - B(x)}\right)A^n(x) - \left(\frac{a_1(x) - A(x)a_0(x)}{A(x) - B(x)}\right)B^n(x).$$

Thus, for the recursion

$$G_n(x) = x^kG_{n-1}(x) + x^kG_{n-2}(x)$$

where $G_0(x) = -1$ and $G_1(x) = x - 1$, we obtain the desired formula. □

This formula can be useful as computing G_n for large values of n via the recursion formula would be rather tedious. However, the brutish nature of this equation can be rather overbearing, so we will avoid using it when other options will be available.

4 Behavior of the Derivatives

In [1], Moore presents the following formuila for $H_n(x)$ relating the value of $H_n(x)$ to the sum of previous indices:

$$H_n(x) = \left(x^2+1\right) H_{n-2}(x) + x^2 \left(\sum_{i=2}^{\lfloor n/2 \rfloor -1} H_{n-2i} \right) + xH_{n-2\lfloor n/2 \rfloor}.$$

We present four different similar formulae for $G_n(x)$ that calculate the value based on only even or only odd previous indices of the sequence of polynomials.

Formulae 3 / 4.

$$G_{2n+1}(x) = (x^{2k}+x^k)G_{2n-1}(x) + \left(\sum_{i=3}^{n} x^{ik} G_{2n+1-2(i-1)}(x) \right) + x^{nk}(G_1(x)-1) \tag{5}$$

$$G_{2n+4}(x) = x^k G_{2n+3}(x) + \left(\sum_{i=2}^{n} x^{ik} G_{2n+5-2(i-1)}(x) \right) + x^{(n+1)k}(G_3(x)+G_1(x)-1) \tag{6}$$

Proof. At first, checking for $n = 2$ we see that

$$\begin{aligned} G_5(x) &= x^k(G_4(x)+G_3(x)) \\ &= x^k[x^k(G_3(x)+G_1(x)-1) + x^k G_3(x)] \\ &= G_3(x)(x^{2k}+x^k) + x^{2k}(G_1(x)-1) \end{aligned}$$

And similarly checking for $G_8(x)$, we have

$$\begin{aligned} G_8(x) &= x^k[G_7(x) + x^k G_5(x) + x^{2k}(G_3(x)+G_1(x)-1)] \\ &= x^k G_7(x) + x^{2k}G_5(x) + x^{3k}(G_3(x)+G_1(x)-1) \end{aligned}$$

So assume this holds for $G_9(x), G_{10}(x), ..., G_{2n}(x)$. Then we have,

$$\begin{aligned} G_{2n+1}(x) &= x^k(G_{2n}(x)+G_{2n-1}(x)) \\ &= x^k[x^k G_{2n-1}(x) + \left(\sum_{i=2}^{n-2} x^{ik} G_{2n+1-2(i-1)}(x) \right) + x^{(n-1)k}(G_3(x)+G_1(x)-1) + G_{2n-1}(x)] \\ &= G_{2n-1}(x)(x^{2k}+x^k) + \left(\sum_{i=2}^{n-2} x^{ik} G_{2n+1-2(i-1)}(x) \right) + x^{nk}(G_3(x)+G_1(x)-1) \\ &= (x^{2k}+x^k)G_{2n-1}(x) + \left(\sum_{i=3}^{n-1} x^{ik} G_{2n+1-2(i-1)}(x) \right) + x^{nk}(G_3(x)+G_1(x)-1) \end{aligned}$$

The proof follows similarly for $G_{2n+4}(x)$ □

Formulae 5 / 6.

$$G_{2n}(x) = (x^{2k}+x^k)G_{2n-2}(x) + \left(\sum_{i=3}^{n-2} x^{ik} G_{2n-2(i-1)}(x) \right) + x^{nk}(x-1) \tag{7}$$

$$G_{2n-1}(x) = x^k G_{2n-2}(x) + \left(\sum_{i=2}^{n-1} x^{ik} G_{2n-2(i-1)}(x) \right) + x^{(n-1)k}(x-1) \tag{8}$$

Proof. Checking for $n = 3$, we see that

$$\begin{aligned} G_6(x) &= x^k(G_5(x) + G_4(x)) \\ &= x^k[x^k G_4(x) + x^{2k} G_2(x) + x^{2k}(x-1) + G_4(x)] \\ &= (x^{2k} + x^k) G_4(x) + x^{3k} G_2(x) + x^{3k}(x-1) \end{aligned}$$

And similarly,

$$\begin{aligned} G_5(x) &= x^k(G_4(x) + G_3(x)) \\ &= x^k G_4(x) + x^{2k} G_2(x) + x^{2k}(x-1) \end{aligned}$$

Assume this holds for $G_7(x), G_8(x), ..., G_{2n-2}(x)$. Then we have

$$\begin{aligned} G_{2n}(x) &= x^k(G_{2n-1}(x) + G_{2n-2}(x)) \\ &= x^k[x^k G2n-2(x) + \left(\sum_{i=2}^{n-1} x^{ik} G_{2n-2(i-1)}(x)\right) + x^{(n-1)k}(x-1) + G_{2n-2}(x) \\ &= (x^{2k} + x^k) G_{2n-2}(x) + \left(\sum_{i=3}^{n-2} x^{ik} G_{2n-2(i-1)}(x)\right) + x^{nk}(x-1) \end{aligned}$$

The proof follows similarly for $G_{2n-1}(x)$ □

This allows us to prove an increasing property of the sequence of derivatives.

Lemma 1.

$$G'_{2n}(x) > G'_{2n-2}(x) > G'_{2n-4}(x) > ... > G'_0(x) = 0$$

for $x \in (g_2, \infty)$ where g_n is the maximal root of $G_n(x)$

Proof. Note that it is easy to see $G'_0(x) = 0$ and $G'_2(x) > 0$ for $x > g_2$. Thus, assume that $G'_{2i}(x) > 0$ for all $i < n$ and $x > g_2$. Then we have

$$\begin{aligned} G'_{2n}(x) &= G'_{2n-2}(x)(x^{2k} + x^k) + (2kx^{2k-1} + kx^{k-1}) G_{2n-2}(x) \\ &\quad + \left(\sum_{i=3}^{n-2} ikx^{ik-1} G_{2n-2(i-1)}(x) + x^{ik} G'_{2n-2(i-1)}(x)\right) + nkx^{nk-1}(x-1) \\ &> \left(\sum_{i=3}^{n-2} G'_{2n-2(i-1)}(x)\right) + G'_{2n-2}(x) \\ &> G'_{2n-2}(x) \end{aligned}$$

□

Lemma 2.

$$G'_{2n+1}(x) > G'_{2n-1}(x) > G'_{2n-3}(x) > ... > G'_1(x) = 1$$

for $x \in (g_{2n-1}, \infty)$ where g_n is the maximal root of $G_n(x)$

Proof. Note that $G_1'(x) = 1$ and $G_3'(x) > 0$ for $x > g_1$. Assume that $G_{2i+1}'(x) > 1$ for all $i < n$ and $x > g_{2i-1}$. Then we have

$$\begin{aligned}
G_{2n+1}'(x) &= G_{2n-1}'(x)(x^{2k} + x^k) + (2kx^{2k-1} + kx^{k-1})G_{2n-1}(x) \\
&\quad + \left(\sum_{i=3}^{n-1} ikx^{ik-1}G_{2n+1-2(i-1)}(x) + x^{ik}G_{2n+1-2(i-1)}'(x)\right) \\
&\quad + nkx^{nk-1}(G_3(x) + G_1(x) - 1) + x^{nk}(G_3(x) + G_1(x) - 1) \\
&> \left(\sum_{i=3}^{n-1} G_{2n+1-2(i-1)}'(x)\right) + G_{2n-1}'(x) \\
&> G_{2n-1}'(x)
\end{aligned}$$

Thus, by interchanging n, we obtain the desired inequality. □

5 Maximal Roots

As stated earlier, computation suggests that the maximal root converges to 2 and that the only positive roots are 0 and the maximal root near 2. To show this, we will be first showing that the root exists by the intermediate value theorem and then that that root converges to 2. We proceed with the following two lemmas.

Lemma 3. $G_n(1) = -F_{n-1}$ for all $n \geq 1$, where F_n is the nth Fibonacci number.

Proof. First, note that $G_1(1) = 0 = -F_0$ and $G_2(1) = -1 = -F_1$. Then by induction let $n \in \mathbb{N}$ such that $G_i(1) = -F_{i-1}$ for all $i \leq n$. Then it follows that

$$G_{n+1}(1) = G_n(1) + G_{n-1}(1) = -F_{n-1} - F_{n-2} = -F_n.$$

□

Lemma 4. $G_n(2) > 0$ for all $n \geq 3$.

Proof.

$$G_3(x) = x^k \left(x - 1 + x^{1+k} - 2x^k\right)$$

and

$$G_4(x) = x^{2k} \left(2x - 3 + x^{1+k} - 2x^k\right)$$

thus $G_3(2) = 2^k > 0$ and $G_4(2) = 2^{2k} > 0$. Now by induction let $n \in \mathbb{N}$ such that $G_i(2) > 0$ for all $i \leq n$. Then $G_{n+1}(2) = 2^k \left(G_n(2) + G_{n-1}(2)\right) > 0$. □

It should also be noted that $G_2(2) = 0$, since $G_2(x) = x^k \left(x - 2\right)$.

Since we now now that $G_n(1) < 0$ and $G_n(2) > 0$ for all $n \geq 3$, it follows by the intermediate value theorem that there is at least one root in the interval $(1, 2)$. We now use the following lemma to show that there are no greater roots than 2.

Lemma 5. $G_n(x) > 0$ for all $n \geq 3$.

Proof. $G_1(x) = x-1$ and $G_2(x) = x^k(x-2)$ are both strictly positive for $x > 2$. Now let $n \in \mathbb{N}$ such that $G_i(x) > 0$ for all $i \leq n$ and $x > 2$. Then $G_n(x) = x^k(G_{n-1}(x) + G_{n-2}(x))$, so for any $x > 2$ it follows that $G_n(x)$ is the product of a positive number with the sum of two positive numbers, thus $G_n(x)$ is also positive for all $x > 2$. □

Thus we have the immediate corrolary:

Corollary 6. There is a maximal root of G_n in the interval $(1, 2)$.

It remains so show that these maximal roots converge to 2 as $n, k \to \infty$. First, we will consider the sequence a_k where a_k denotes the maximal root of the equation

$$P_k(x) = x^{k+1} - 2x^k + x^2 - 2x + 1.$$

Lemma 7. The sequence a_k converges to 2 as k approaches infinity.

Proof. Let $Q_k(x) = P(2-x) = 1-(2-x)^k x - (2-x)x$. Then it remains to show that the minimal root ϵ_k of $Q_k(x)$ converges to 0. First, note that $Q_k(x) > 0$ for all $x < 0$. Then,

$$\begin{aligned}
& Q_k\left(\frac{1}{2^n}\right) < 0 \\
\iff\ & 1 - \left(\frac{2^{n+1}-1}{2^n}\right)^k \frac{1}{2^n} - \left(\frac{2^{n+1}-1}{2^n}\right)\frac{1}{2^n} < 0 \\
\iff\ & 2^n < \left(\frac{2^{n+1}-1}{2^n}\right)\left(\left(\frac{2^{n+1}-1}{2^n}\right)^{k-1} + 1\right) \\
\iff\ & 2^n A_n^{-1} - 1 < A_n^{k-1} \text{ where } A_n = \left(\frac{2^{n+1}-1}{2^n}\right) \\
\Longleftarrow\ & 2^n A_n^{-1} < A_n^{k-1} \\
\iff\ & \log_2\left(2^n A_n\right) < (k-1)\log_2\left(A_n\right) \\
\iff\ & n + \log_2\left(A_n\right) < (k-1)\log_2\left(A_n\right) \\
\Longleftarrow\ & n + 1 < (k-1)\frac{1}{2} \text{ (Since } n \geq 1) \\
\iff\ & k > 2n + 3
\end{aligned}$$

Thus $\epsilon_k < \frac{1}{2^n} \iff k > 2n+3$. Accordingly, for any $\epsilon > 0$, let M be any natural number that satisfies $\frac{1}{2^M} < \epsilon$. Then, letting $N = 2M + 3$, it follows that $\epsilon_k < \epsilon$ for all $k > N$. Thus $\lim_{k\to\infty} \epsilon_k = 0$ and $\lim_{k\to\infty} a_k = 2$. □

The following formula will also prove useful:

Formula 7.

$$G_n(a_k) = -(1-a_k)^n \tag{9}$$

Proof. First, so that $G_0(a_k) = -1 = -(1-a_k)^0$ and $G_1(a_k) = a_k - 1 = -(1-a_k)^1$. Then suppose that $G_i(a_k) = -(1-a_k)^i$ for all $i < n$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} G_n(a_k) &= a_k^k \left(G_{n-1}(a_k) + G_{n-2}(a_k)\right) \\ &= a_k^k \left(-(1-a_k)^{n-1} - (1-a_k)^{n-2}\right) \\ &= a_k^k \left(-(1-a_k)^{n-2}\right)(1-a_k+1) \\ &= -(1-a_k)^{n-2}\left(2a_k^k - a_k^{k+1}\right) \\ &= -(1-a_k)^{n-2}(a_k-1)^2 \\ &= -(1-a_k)^n \end{aligned}$$

□

Next, we will establish a formula for G_n that we will denote the shift-formula.

Formula 8.

$$G_{n+r}(g_n) = (-1)^{r+1} g_n^{rk}\ G_{n-r}(g_n) \tag{10}$$

Proof. First, for $r = 1$ we see that

$$G_{n+1}(x) = g_n^k \left[G_n(g_n) + G_{n-1}(g_n)\right] = g_n^k G_{n-1}(g_n)$$

and for $r = 2$

$$G_{n+2}(g_n) = g_n^k \left[G_{n+1}(g_n) + G_n(g_n)\right] = g_n^k \left[G_{n+1}(g_n)\right] = g_n^{2k}\left[G_{n-1}(g_n)\right].$$

Now since $G_n(g_n) = G_{n-1}(g_n) + G_{n-2}(g_n)$, then $G_{n-1}(g_n) = -G_{n-2}(g_n)$. Thus $G_{n+2}(g_n) = -g_n^{2k} G_{n-2}(g_n)$. Now let $j \in \mathbb{N}$ such that $G_{n+r}(g_n) = (-1)^{r+1} g_n^{rk}\ G_{n-r}(g_n)$ for all $r < j$. Then let

$$A = \frac{G_{n+j}(g_n)}{G_{n-j}(g_n)} = g_n^{2k} \frac{G_{n+j-1}(g_n) + G_{n+j-2}(g_n)}{G_{n-j+2}(g_n) - g_n^k G_{n-j+1}(g_n)},$$

$$B = G_{n+(j-1)}(g_n) = (-1)^j g_n^{(j-2)k} G_{n-(j-1)}(g_n), \text{ and}$$

$$C = G_{n+(j-2)}(g_n) = (-1)^{j+1} g_n^{(j-2)k} G_{n-(j-2)}(g_n).$$

Then by substitution we have

$$A = \frac{g_n^{2k}(B+C)}{C\frac{(-1)^{j-1}}{g_n^{(j-2)k}} - B\frac{g_n^k}{g_n^{(j-1)k}}} = \frac{g_n^{2k} g_n^{jk-2k}(B+C)}{(-1)^{j-1}C - B(-1)^j} = \frac{g_n^{jk}}{(-1)^{j-1}} = g_n^{jk}(-1)^{j+1}.$$

□

We will now begin the process of showing that the sequence g_{2n} and g_{2n+1} both converge to the same value.

Lemma 8. $g_{2n+1} < a_k < g_{2n}$ for all $n \in \mathbb{N}$

Proof. For any n, it follows that $G_{2n}(a_k) = -(1-a_k)^{2n} < 0$, thus $g_{2n} > a_k$.
Using formula [3],

$$G_{2n+1}(x) = \left(x^{2k} + x^k\right) G_{2n-1}(x) + \sum_{i=3}^{n} x^{ik} G_{2n+3-2i}(x) + x^{nk} G_1(x) - x^{nk}.$$

Thus, since $a_k > g_1 = 1$, assume that for all $i < n$ that $g_{2i+1} < a_k$. Then, since x^{jk} has no positive roots[1] for any $j \in \mathbb{N}$, it follows that $g_{2n+1} < a_k$. □

The shift formula will be useful for the following lemma:

Lemma 9. Letting g_n denote the maximal root of G_n, the sequence (g_{2n}) is monotonically decreasing for $n \geq 0$ and the sequence (g_{2n+1}) is monotonically increasing.

Proof. Since $g_2 = 2$ and we know $g_4 < 2$, we know that $g_4 < g_2$. Similarly, we know $g_1 = 1$ and that $g_3 > 1$, so $g_3 > g_1$. Now let $n \in \mathbb{N}$ such that $g_{2i+2} > g_{2i}$ for all $i < n$. Then $G_{2n+2}\left(g_{2n}\right) = -g_{2n}^{2k} G_{2n-2}\left(g_{2n}\right)$. Thus $g_{2n-2} > g_{2n}$

Further, let $n \in \mathbb{N}$ such that $g_{2i+1} > g_{2i-1}$ for all $i < n$. Then $G_{2n+1}\left(g_{2n-1}\right) = -g_{2n-1}^{2k} G_{2n-3}\left(g_{2n-1}\right)$. Since $g_{2n-3} < g_{2n-1}$, it follows that $G_{2n-3}\left(g_{2n-1}\right) > 0$. Thus $G_{2n+1}\left(g_{2n-1}\right) < 0$ and finally $g_{2n+1} > g_{2n-1}$. □

Lemma 10.

$$G'_{2n+4}(x) > 1$$

for $x \in (a_k, \infty)$

Proof. Note that

$$\begin{aligned}
G'_{2n+4} &= kx^{k-1} G_{2n+3}(x) + x^k G'_{2n+3}(x) + \left(\sum_{i=2}^{n} ikx^{ik-1} G_{2n+5-2(i-1)}(x)\right) \\
&\quad + (n+1)kx^{(n+1)k-1}(G_3(x) + G_1(x) - 1) + x^{(n+1)k}(G'_3(x) + G'_1(x)) \\
&> x^k G'_{2n+3}(x) + \left(\sum_{i=2}^{n} x^{ik} G'_{2n+5-2(i-1)}(x)\right) + x^{(n+1)k}(G'_3(x) + G'_1(x)) \\
&> \sum_{i=1}^{n+1} x^{ik} \\
&> 1
\end{aligned}$$

□

Lemma 11.

$$\lim_{n \to \infty} g_{2n} = a_k$$

for $x \in (a_k, \infty)$

Proof. Let $f(x) = G_{2n}(x)$, and $h(x) = x - (a_k + (1-a_k)^{2n})$. Now, note that $f(a_k) = h(a_k) < 0$. And $f'(x) > h'(x) = 1$. Thus $g(x)$ has a root at $x = (a_k + (1-a_k)^{2n})$, which implies that $f(x)$ has a root at $(a_k, a_k + (1-a_k)^{2n})$. Hence $a_k < g_{2n} < a_k + (1-a_k)^{2n}$ as $n \to \infty$ □

[1] For more information on this, see Moore's paper

Lemma 12.

$$\lim_{n\to\infty} g_{2n+1} = a_k$$

for $x \in (g_{2n+1}, \infty)$

Proof. Let $f(x) = G_{2n+1}(x)$, and $h(x) = x-(a_k+(1-a_k)^{2n+1})$. Now, note that $f(a_k) = h(a_k) > 0$. And $f'(x) > h'(x) = 1$. Thus $g(x)$ has a root at $x = (a_k + (1-a_k)^{2n+1})$, which implies that $f(x)$ has a root at $(a_k + (1-a_k)^{2n+1}, a_k)$. Hence $a_k > g_{2n} > a_k + (1-a_k)^{2n}$ as $n \to \infty$ □

6 Generating Function

Given a sequence a_n, the generating function for that sequence is $\sum_{n=0}^{\infty} a_n x^n$. The generating function for the sequence G_n is

$$\sum_{n=0}^{\infty} G_n(x)x^n = \frac{x^{k+1} + x^2 - (x+1)}{1 - x^{k+1}(1+x)}. \tag{11}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n(x)x^n &= \frac{\left(1 - x^{k+1} - x^{k+2}\right)}{\left(1 - x^{k+1} - x^{k+2}\right)} \sum_{n=0}^{\infty} G_n(x)x^n \\
&= \frac{\sum_{n=0}^{\infty} G_n(x)x^n - \sum_{n=1}^{\infty} G_{n-1}(x)x^n x^k - \sum_{n=2}^{\infty} G_{n-2}x^n x^k}{1 - x^{k+1} - x^{k+2}} \\
&= \frac{(G_0(x) + xG_1(x)) - \left(G_0(x)x^{k+1}\right)}{1 - x^{k+1} - x^{k+2}} + \frac{\sum_{n=2}^{\infty} \left(G_n(x) - x^k\left(G_{n-1}(x) + G_{n-2}(x)\right)\right)}{1 - x^{k+1} - x^{k+2}} \\
&= \frac{x^{k+1} + x^2 - x - 1}{1 - x^{k+1} - x^{k+2}}
\end{aligned}$$

□

7 Coefficients

Up to this point, we have only shown that there exists a root in the interval $(1, 2)$, and not that there exists only one root. To show this, we wanted to apply Budan's Theorem, which is stated as follows:

Theorem 1. Budan's Theorem Given a polynomial $p(x)$ and an interval (l, r), let (α_n) be the non-zero coefficients in order of power of $p(x + l)$ and let (β_n) be the non-zero coefficients in order of power of $p(x+r)$. If there is exactly one less sign change in (β_n) than (α_n), then there is exactly one root of $p(x)$ in the interval (l, r).

To accomplish this, we first considered $G_n(x + 2)$.

Lemma 10. The coefficients $\{\beta_i\}$ of $G_n(x + 2) = \sum_{i=0}^{N} \beta_i x^i$ where N is the degree of G_n are all non-negative for $n \geq 1$.

Proof. First, note that $G_1(x + 2) = x + 1$, so the coefficients are all $1 \geq 0$ and

$$G_2(x + 2) = (x + 2)^k(G_1(x + 2) + G_0(x + 2)) = (x + 2)^k(x + 1 - 1) = (x + 2)^k x$$

which has all non negative coefficients. Now assume that for all $i \leq n$ all coefficients of $G_i(x+2)$ are non negative. Then $G_{n+1}(x+2) = (x+2)^k(G_n(x+2) + G_{n-1}(x+2))$. Since $(x+2)^k$ and both $G_n(x+2)$ and $G_{n-1}(x+2)$ have only non negative coefficients, it follows that $G_{n+1}(x+2)$ will only have non negative coefficients. □

Thus, for Budan's Theorem to apply, we would need to show that there exists exactly one sign change in the coefficients of $G_n(x+1)$. However, this can be difficult without an exact formula for the coefficients. Induction did not seem applicable because even if you have that $G_{n-1}(x+1)$ and $G_{n-2}(x+1)$ each have exactly one sign change in their coeffients, there doesn't seem to be an accessible reasoning that there would be exactly one sign change in $G_n(x+1)$. Thus we need a formula for the exact coefficients. Before this, we will briefly discuss the multi-level triangle numbers.

The triangle numbers (1, 3, 6, 10, 15, ...) are a well known sequence of numbers formed by summing up the first n natural numbers. That is, if $T(n)$ denotes the nth triangle number, then

$$T(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

This can be generalized by defining $T_i(n) = \sum_{i=1}^{n} T_{i-1}(n)$, where $T_0(n) = 1$. We call $T_m(n)$ the nth number in the sequence of the mth level triangle numbers. These numbers can be thought of as the triangle numbers in multiple dimensions, as T_1 is equivalent to counting the number of dots in a line, T_2 forms the regular triangle numbers, and T_3 forms a sequence generally known as the tetrahedral numbers. It is known that $T_m(n) = \binom{n+m-1}{m}$.

With that, we procede with the formula:

Formula 10. For $n \geq 2$

$$G_n(x) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \Big[T_i(n-2i)x - (T_i(n-2i) + T_i(n-2i-1)) \Big] x^{(n-i-1)k} \tag{12}$$

Proof. First, for $n = 2$, we have

$$\sum_{i=0}^{0} \Big[T_i(2-2i)x - (T_i(2-2i) + T_i(2-2i-1)) \Big] x^{(2-i-1)k} = (x-2)x^k = G_2(x).$$

Similarly, for $n = 3$, we have

$$\begin{aligned} &\sum_{i=0}^{1} \Big[T_i(3-2i)x - (T_i(3-2i) + T_i(3-2i-1)) \Big] x^{(3-i-1)k} \\ &= \Big[T_0(3)x - (T_0(3) + T_0(2)) \Big] x^{2k} + (T_1(1)x - (T_1(1) + T_1(0)))x^k = G_3(x) \end{aligned}$$

Now suppose that for some $n > 2$ the above formula holds for G_{n-1} and G_{n-2}. Then

$$\begin{aligned}
G_n(x) &= x^k \left(G_{n-1}(x) + G_{n-2}(x)\right) \\
&= x^k \left(\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil - 1} \Big[T_i(n-2i-1)x - (T_i(n-2i-1) + T_i(n-2i-2)) \Big] x^{(n-i-2)k} \right. \\
&\quad \left. + \sum_{i=0}^{\lceil \frac{n-2}{2} \rceil - 1} \Big[T_i(n-2i-2)x - (T_i(n-2i-2) + T_i(n-2i-3)) \Big] x^{(n-i-3)k} \right) \\
&= x^k \left(\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil - 1} \Big[T_i(n-2i-1)x - (T_i(n-2i-1) + T_i(n-2i-2)) \Big] x^{(n-i-2)k} \right. \\
&\quad \left. + \sum_{i=1}^{\lceil \frac{n-2}{2} \rceil} \Big[T_{i-1}(n-2i)x - (T_{i-1}(n-2i) + T_{i-1}(n-2i-1)) \Big] x^{(n-i-2)k} \right) \\
&= x^k \left((x-2)x^{(n-2)k} + \sum_{i=\lfloor \frac{n}{2} \rfloor - 1}^{\lceil \frac{n}{2} \rceil - 1} \Big[T_{i-1}(n-2i)x - (T_{i-1}(n-2i) \right. \\
&\quad + T_{i-1}(n-2i-1)) \Big] x^{(n-i-2)k} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} \Big[\left(T_i(n-2i-1) + T_{i-1}(n-2i)\right) x \\
&\quad \left. - \left(T_i(n-2i-1) + T_i(n-2i-2) + T_{i-1}(n-2i) + T_{i-1}(n-2i-1)\right) \Big] x^{(n-i-2)k} \right) \\
&= x^k \left(\sum_{i=\lfloor \frac{n}{2} \rfloor - 1}^{\lceil \frac{n}{2} \rceil - 1} \Big[T_{i-1}(n-2i)x - (T_{i-1}(n-2i) + T_{i-1}(n-2i-1) \Big] x^{(n-i-2)k} \right. \\
&\quad \left. + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \Big[T_i(n-2i)x - (T_i(n-2i) + T_i(n-2i-1)) \Big] x^{(n-i-2)k} \right) \\
&= \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \Big[T_i(n-2i)x - (T_i(n-2i) + T_i(n-2i-1)) \Big] x^{(n-i-1)k}
\end{aligned}$$

□

This formula will allow us to plug in $x+1$ for x and expand via the standard binomial expansion rules where applicable:

$$
\begin{aligned}
G_n(x+1) &= \sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} \Big[T_i(n-2i)x + T_i(n-2i) - (T_i(n-2i) + T_i(n-2i-1))\Big](x+1)^{(n-i-1)k} \\
&= \sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} \sum_{r=0}^{(n-i-1)k} \Big[T_i(n-2i)x - T_i(n-2i-1)\Big]\binom{(n-i-1)k}{r} x^r \\
&= \sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} \Bigg[-T_i(n-2i-1) + T_i(n-2i) \sum_{r=0}^{(n-i-1)k-1} \binom{(n-i-1)k}{r} x^{r+1} \\
&\quad - T_i(n-2i-1) \sum_{r=1}^{(n-i-1)k} \binom{(n-i-1)k}{r} x^r + T_i(n-2i)x^{(n-i-1)k+1} \Bigg] \\
&= \sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} \Bigg[-T_i(n-2i-1) + T_i(n-2i)x^{(n-i-1)k+1} \\
&\quad + \sum_{r=1}^{(n-i-1)k} \left(T_i(n-2i)\binom{(n-i-1)k}{r-1} - T_i(n-2i-1)\binom{(n-i-1)k}{r} \right) x^r \Bigg] \\
&= -\sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} T_i(n-2i-1) + \sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} T_i(n-2i)x^{(n-i-1)k+1} \\
&\quad + \sum_{i=0}^{\lceil \frac{n}{2}\rceil -1} \sum_{r=1}^{(n-i-1)k} \left(T_i(n-2i)\binom{(n-i-1)k}{r-1} - T_i(n-2i-1)\binom{(n-i-1)k}{r} \right) x^r
\end{aligned}
$$

This results in a formula that gives us an explicit formula for the coefficients. However, work remains to be done to show that there is exactly one sign change.

8 Further Work

In this paper we presented various analytic results about the sequence of polynomials G_n where $\ell = k$ as in equation 1. We believe many of these results to be generalizable to the cases where $\ell \neq k$. Further work might generalize the location of the maximal roots in these cases. Specifically for the case where $\ell = k$, using the final equation for $G_n(x+1)$ it would be interesting to confirm that there is exactly one sign change among the coefficients - this can easily be confirmed via Mathematica for small enough n.

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References

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