Research Projects for SURIEM 2016

1. Combinatorial games on graphs. Consider the following two player, complete information game of Cops and Robber played on a finite combinatorial graph G. The first player controls a fixed number of cops and she assigns each to a vertex of G. The second player controls a single robber; after the cops are placed, the robber is placed on a vertex. Players alternate turns and move any number of their pawns to adjacent vertices on his or her turn. The cops win if the robber is captured by moving a cop to the same vertex; the robber wins if this outcome can be avoided. The fundamental problem is to determine the fewest number of cops, the cop number of G, denoted c(G), so that the cops can always win. Of recent interest is the weak cop number of an infinite graph: the fewest number of cops needed to prevent the robber from visiting any vertex infinitely often.

The following are sample projects REU participants can work on.

- Which theorems on the cop number of finite graphs remain true for the weak cop number of infinite graphs? For instance, the students in the 2015 REU proved that a 3-regular graph of girth greater than 3 need not have a weak cop number greater than 2; an example is the hexagonal honeycomb.
- Containment is a variant of cops and robbers in which cops occupy edges and move to incident edges so as to surround a robber who occupies vertices and moves to adjacent vertices. Investigate the game of weak containment where cops try to prevent the robber from visiting any vertex infinitely often.

2. Recursive polynomials. Consider a Fibonacci-type polynomial sequence given by the recurrence relation

\[ G_0(x) = \alpha, \quad G_1(x) = x + \beta, \quad G_n(x) = \gamma(x)G_{n-1}(x) + G_{n-2}(x), \quad n \geq 2. \]

Here \(\alpha\) and \(\beta\) are integers and \(\gamma\) is some function of \(x\). If \(\alpha = 1, \beta = 0,\) and \(\gamma(x) = x,\) then \(G_n(x)\) is the classical Fibonacci polynomial sequence \(F_n(x)\). For \(\alpha = 2, \beta = 0,\) and \(\gamma(x) = x\) one gets the Lucas polynomial sequence \(L_n(x)\).

The following are sample projects for REU participants.

- Find the asymptotic behavior of the roots of the \(k\)-th derivative of the Fibonacci-type polynomial sequence \(G_0(x) = \alpha, G_1(x) = \ldots\)
and \( G_n(x) = x^k G_{n-1}(x) + G_{n-2}(x), \ n \geq 1 \). How does the order of the derivative impact the behavior of the maximum roots?

• What is the asymptotic behavior of the roots of other Fibonacci-type polynomials? For example a Fibonacci-type polynomial sequence generated using three initial conditions? For what other \( \gamma \)'s and initial conditions can one get results of the same nature?

3. **Random fractals and Brownian motion.** Brownian motion is a powerful stochastic model that has been applied in many scientific fields, from physics to finance, to biological sciences. The same can be said about fractal geometry. Since Brownian motion is not differentiable and generates various interesting fractal sets and measures, it is natural (and necessary) to apply tools from fractal geometry (e.g. Hausdorff dimension, packing dimension) to study Brownian motion. Even though there is an enormous literature on sample path properties of Brownian motion), properties in terms of the discrete Hausdorff dimension and packing dimension introduced by Barlow and Taylor have not been investigated. The objective of this research group is to investigate such properties of Brownian motion and other related stochastic processes. It is expected that the large scale fractal properties of a stochastic process rely on asymptotic properties of the process at infinity, which may be different from the local properties.

The following are sample research projects for REU participants.

• Let \( B = \{B(t), t \geq 0\} \) be Brownian motion with values in \( \mathbb{R}^d \), let

\[
\text{Gr} B(\mathbb{R}) = \{(t, B(t)) \mid t \geq 0\}
\]

be its graph set, and let

\[
M_k = \{x \in \mathbb{R}^d \mid \exists \ \text{distinct} \ t_1, \ldots, t_k \in B^{-1}(x)\}
\]

be the set of \( k \)-multiple points. Determine the discrete Hausdorff and packing dimensions of \( \text{Gr} B(\mathbb{R}) \) and \( M_k \).

• Let \( X = \{X(t), t \in \mathbb{R}^N\} \) be a Gaussian random field with values in \( \mathbb{R}^d \), which have long range dependence (Hurst effect) and local fractal properties. Investigate the discrete Hausdorff and packing dimensions of the random sets generated by \( X \).
4. Mathematics of Magnetic Resonance Imaging (MRI)

Since its introduction in the 1970s, magnetic resonance imaging\(^1\) (MRI) has become an invaluable (and non-invasive) diagnostic tool in medicine. MR imaging incorporates several fascinating ideas from physics, engineering, statistics and computer science. In this REU, we will explore the mathematical underpinnings of MR image reconstruction. Over the course of the program, we will explore

(a) the basic physics of MR imaging leading to a mathematical formulation of the image reconstruction process,

(b) the data acquisition process in MR scanners, including how scanners sample data, the number of samples acquired and trade-offs between reconstruction accuracy and scan time,

(c) different reconstruction procedures and their relative advantages and disadvantages vis-à-vis reconstruction accuracy, computational cost and robustness to measurement errors,

(d) new developments and cutting-edge research ideas aimed at reducing data acquisition times and improving MR image quality.

Throughout the program, we will work with both simulated and actual MRI data and write computer code simulating the MR image recon-

\(^1\)Picture credits: Jan Ainali (MR scanner) and Kieran Maher (MRI scans).
struction process. By the end of the program, in addition to a working knowledge of the basics of MRI, you will also have been introduced to selected and relevant topics in Fourier analysis, signal processing and sampling theory, probability, linear algebra and numerical methods.

The following are some sample projects for REU participants:

(a) *Accelerating Parallel MRI Reconstruction*: Parallel MRI is a variant of conventional imaging where multiple simultaneous observations of the specimen are observed, leading to reduced scan times. We will explore new reconstruction strategies aimed at improving reconstruction accuracy and computational cost while exploiting underlying structure in the scan images.

(b) *Compressed Sensing MR Imaging*: Compressed sensing MRI dramatically reduces scan times by collecting significantly fewer data samples than conventional imaging methods. The price paid for this is a more complicated (and non-linear) reconstruction procedure which exploits known structure in MR scans; specifically, that scan images are “sparse” in some representation\(^2\). We will study such imaging modalities and develop new faster, more accurate and computationally efficient reconstruction schemes.

(c) *Reconstruction of MRI Images from Non-Uniform Samples*: We will consider image reconstruction from non-uniformly sampled data. Such data acquisitions are known be more robust to measurement errors, even though the reconstruction procedure is more complicated. We will explore fundamental limits in the types of non-uniform samples which permit accurate reconstruction and develop improved numerical procedures for imaging from such data.

\(^2\)for example, wavelet representations (similar to those used in JPEG image compression).